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A HYDRAULIC FRACTURE IN A TRANSVERSELY-ISOTROPIC POROELASTIC MEDIUM†

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The problems of plane and axisymmetric disk-shaped cracks in hydraulic fracture [1, 2] are considered in transversely-isotropic poroelastic media saturated with fluid. The crack is opened up by a flux of viscous fluid filtering through a stratum. The stressed state and deformation of the poroelastic media is described by the Biot equations [3]. An analytic solution is found for a stationary "ideal" disk-shaped crack along which the pressure is constant.

In classical hydraulic fracture theory [1, 2, 4, 5] for an isotropic stratum, effects associated with poroelasticity have been ignored in the description of crack propagation [6–9]. However, poroelastic effects are important in most problems of practical interest [10–12]. Moreover, the actual media in which hydraulic fracture occurs are, as a rule, anisotropic.

Hydraulic fracture problems for an isotropic poroelastic medium have been considered previously [10–18]. This paper develops a method for solving hydraulic fracture problems for transversely-isotropic media. This, together with the results obtained, generalize the approach and some of the results of [18].

1. STATEMENT OF THE PROBLEM

Suppose there is a plane (axisymmetric) crack in an infinite transversely-isotropic porous space saturated with fluid and that there is a homogeneous compressive stress field σ_∞ supported in an open state by fluid injected into it. The injected fluid moves along the crack (radially) and can filter through its walls into the porous space. It is assumed that the plane crack is perpendicular to the axis of symmetry x_2, and the x_1 axis is directed along (lies in the plane) of the crack. It is moreover assumed that the radius of the borehole r_0 is much smaller than the crack length L, so that effects due to the borehole can be ignored.

To describe the deformation of the transversely-isotropic porous medium saturated with fluid we use the Biot equations for coupled consolidation [3] ($i, j, k = 1, 2, 3$; summation is carried out over repeated indices)

$$\nabla_j \sigma_{ij} = 0, \quad \sigma_{ij} = \sigma_{ji} \tag{1.1}$$

$$\left\| \begin{matrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \dfrac{E}{2(1-\nu)\rho_0}[m-m_0] \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{1}{E} & -\dfrac{\nu'}{E'} & -\dfrac{\nu}{E} & 0 & \dfrac{2(1-\nu)\eta}{E'} \\ -\dfrac{\nu'}{E'} & \dfrac{1}{E'} & -\dfrac{\nu'}{E'} & 0 & \dfrac{2\eta(1-\nu)}{E'}\eta' \\ -\dfrac{\nu}{E} & -\dfrac{\nu'}{E'} & \dfrac{1}{E} & 0 & \dfrac{2(1-\nu)\eta}{E} \\ 0 & 0 & 0 & \dfrac{1}{G'} & 0 \\ \eta & \dfrac{E}{E'}\eta' & \eta & 0 & \dfrac{1}{B}\left(2\eta+\dfrac{E}{E'}\eta'\right) \end{matrix} \right\| \left\| \begin{matrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ p \end{matrix} \right\| \tag{1.2}$$

$$\frac{\partial}{\partial t} m + \nabla_i(\rho_0 u_i) = 0 \tag{1.3}$$

$$u_1 = -\frac{k_1 \delta_{li}}{\mu} \frac{\partial}{\partial x_i} p, \quad k_1 = k_3 \neq k_2 \tag{1.4}$$

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Here (1.1) is the equilibrium equation, (1.2) are governing relations, (1.3) are the continuity equations for a filtering fluid, and (1.4) is D'Arcy's law for a transversally-isotropic porous medium, σ_{ij} is the total stress tensor, ε_{ij} is the strain tensor, E and E' are Young's moduli, and ν and ν' are Poisson's ratios for a poroelastic medium, G and G' ($G = E/([2(1+\nu)])$) are shear moduli, p is the pore pressure, m is the mass of permeating fluid per unit volume, m_0 and ρ_0 are the mass and density of the permeating fluid in the undeformed state, k_1, k_3 and k_2 are the permeability coefficients in the plane perpendicular to the axis of symmetry and in the direction of the axis of symmetry, μ is the viscosity of the fluid, u_i is the filtration rate in the ith direction, and δ_{ij} is the Kronecker delta.

The parameters η and η' can be expressed in terms of the parameters α and α' introduced in [14]

$$\eta = \frac{1}{2}\left(\alpha - \frac{\nu' E}{(1-\nu)E'}\alpha'\right), \quad \eta' = \frac{1}{2(1-\nu)}(\alpha' - 2\nu'\alpha)$$

$$\alpha = \{1 - \lambda_1[\lambda_2 + \nu'\lambda_3]\}/(2b+b'), \quad \alpha' = \{1-\lambda_1[2\nu''\lambda_2 + (1-\nu)\lambda_3]\}/(2b+b')$$

$$\lambda_1 = \frac{E}{1-\nu-2\nu'\nu''}, \quad \lambda_2 = \frac{1-\nu_u - \nu_u''}{E_u}, \quad \lambda_3 = \frac{1-2\nu_u'}{E_u'}$$

$$\frac{\nu_u''}{E_u} = \frac{\nu_u'}{E_u'}, \quad \frac{\nu''}{E_u} = \frac{\nu'}{E'}, \quad b = \frac{(1-A)B}{2}, \quad b' = AB$$

where E_u and E_u' are Young's moduli and ν_u and ν_u' are Poisson's ratios appropriate to the condition that the fluid cannot leave the medium, and A and B are the Skempton parameters.

We will describe the motion of the injected fluid along the crack by the continuity equation and Poiseuille's law

$$\frac{\partial}{\partial t}w + \frac{1}{x_1^n}\frac{\partial}{\partial x_1}(x_1^n w u) = -2\upsilon_L, \qquad u = -\frac{w^2}{12\mu}\frac{\partial}{\partial x_1}p_c \tag{1.5}$$

where w is the size of the opening between the crack edges, p_c is the pressure of the fracture fluid injected into the crack, υ_L is the rate of leakage of the fracture fluid into the stratum across the crack walls and n is the symmetry index for the problem ($n = 0$ for a plane crack and $n = 1$ for an axisymmetric crack).

At the crack edges we impose the following boundary conditions

$$p_c(x_1,t) = p(x_1, x_2 = 0, t) \tag{1.6}$$

$$\upsilon_L(x_1,t) = -\frac{k_2}{\mu}\frac{\partial}{\partial x_2}p(x_1, x_2 = 0+0, t) \tag{1.7}$$

2. THE PLANE PROBLEM

In the plane strained state ($\varepsilon_{33} = 0$), which will be considered again later, the governing relations (1.2) take the form

$$\left\| \begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \dfrac{E[m-m_0]}{2(1-\nu)\rho_0} \end{array} \right\| = \left\| \begin{array}{cccc} \dfrac{1-\nu^2}{E} & -\dfrac{(1+\nu)\nu'}{E'} & 0 & \dfrac{2\eta(1-\nu^2)}{E} \\ -\dfrac{(1+\nu)\nu'}{E'} & \dfrac{1-\nu'^2\Xi}{E'} & 0 & \dfrac{2\eta(1-\nu)}{E'}\left[\dfrac{\eta'}{\eta}+\nu'\right] \\ 0 & 0 & \dfrac{1}{2G'} & 0 \\ \eta'(1+\nu) & (\eta'+\nu'\eta)\Xi & 0 & \chi \end{array} \right\| \left\| \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ p \end{array} \right\|$$

$$\Xi = E/E', \quad \chi = B^{-1}[2\eta(1-2\eta B(1-\nu)) + \eta'\Xi] \tag{2.1}$$

Here the equilibrium equations (1.1) can be written in x_1 and x_2 coordinates in the form

$$\partial\sigma_{ij}/\partial x_j = 0, \quad \sigma_{ij} = \sigma_{ji} \tag{2.2}$$

In order to satisfy the equilibrium equations (2.2) identically for the plane problem we introduce the

Airy function (α, α' = 1, 2, summation is not performed over repeated Greek indices)

$$\alpha_{\alpha\beta} = (-1)^{\alpha+\beta}\partial^2 F / \partial x_{3-\alpha}\partial x_{3-\beta} \tag{2.3}$$

From the deformation consistency conditions

$$\partial^2\varepsilon_{11} / \partial x_2^2 + \partial^2\varepsilon_{22} / \partial x_1^2 = 2\partial^2\varepsilon_{12} / \partial x_1\partial x_2$$

and the governing relations (2.1) using (2.3) we obtain

$$a_1 \frac{\partial^4}{\partial x_1^4} F + a_2 \frac{\partial^4}{\partial x_1^2\partial x_2^2} F + \frac{\partial^4}{\partial x_2^4} F = -2\eta\left(b\frac{\partial^2}{\partial x_1^2} p + \frac{\partial^2}{\partial x_2^2} p\right) \tag{2.4}$$

$$a_1 = \frac{1-\nu'^2\Xi}{1-\nu^2}\Xi, \quad a_2 = 2\frac{1-\nu'\Xi}{1-\nu}, \quad b = \frac{\eta'/\eta+\nu'}{1+\nu}\Xi$$

To solve Eq. (2.4) we introduce complex variables through the formulae $z_\alpha = x_1 + i\mu_\alpha x_2$ and their complex conjugates $z_\alpha^* = x_1 - i\mu_\alpha x_2$.

Let μ_1 and μ_2 be roots of the equation

$$\mu^4 - a_2\mu^2 + a_1 = 0 \tag{2.5}$$

The left-hand side of Eq. (2.4) can be written in the form

$$16(\mu_1\mu_2)^2 \frac{\partial^4}{\partial z_1\partial z_1^*\partial z_2\partial z_2^*} F$$

In order to obtain a complex representation of the right-hand side of Eq. (2.4) we use properties of the roots of Eq. (2.5). They are either real ($\mu_1 = \mu_1, \mu_2 = \mu_2, \mu_3 = -\mu_1, \mu_4 = -\mu_2$) or complex-conjugate ($\mu_1 = \mu, \mu_2 = \bar{\mu}, \mu_3 = -\mu, \mu_4 = -\mu$). If the roots of Eq. (2.5) are real, then $z_\alpha^* = \bar{z}_\alpha$. If they are complex-conjugate, then $z_\alpha^* = \bar{z}_{3-\alpha}$. Equal roots correspond to isotropic elasticity theory [20], and we do not consider this case. From what has been said, the following complex representation can be obtained for Eq. (2.4)

$$\frac{\partial^4}{\partial z_1\partial z_1^*\partial z_2\partial z_2^*} F = -\frac{1}{2}\eta \sum_{\alpha=1}^{2} \kappa_{3-\alpha} \frac{\partial^2}{\partial z_\alpha\partial z_\alpha^*} p, \quad \kappa_\alpha = \frac{b-\mu_\alpha^2}{\mu_\alpha^2(\mu_{3-\alpha}^2 - \mu_\alpha^2)} \tag{2.6}$$

(The independent variables x_1 and x_2 are either expressed in terms of complex variables z_1 or z_1^* or in terms of z_2, z_2^*.)

To integrate Eq. (2.6) we use the following method [19]. The independent variables x_1, x_2 and the functions F and p are taken to be complex. In this situation the new variables z_1, z_1^* and z_2, z_2^* become independent. At the same time the independent variables z_1, z_1^* can be expressed in terms of z_2, z_2^* and vice-versa. After performing the required calculations one returns to the original variables in which x_1 and x_2 are real, and z_α and z_α^* (α = 1, 2) become conjugate values of a single complex variable.

Integrating Eq. (2.6) and using the fact that in the change to real variables x_1 and x_2 the Euler function F should be real, we obtain

$$F = \sum_{\alpha=1}^{2} \left\{ f_\alpha(z_\alpha) + f_\alpha^*(z_\alpha^*) - \frac{1}{2}\eta\kappa_\alpha \int_{z_{0\alpha}}^{z_\alpha} \int_{z_{0\alpha}^*}^{z_\alpha^*} d\xi_\alpha d\xi_\alpha^* p(\xi_\alpha, \xi_\alpha^*) \right\} \tag{2.7}$$

where $f_1(z_1), f_2(z_2), f_1^*(z_1^*), f_2^*(z_2^*)$ are analytic functions and $z_{01}, z_{01}^*, z_{02}, z_{02}^*$ are constants.

Using the fact that in the change to real variables x_1 and x_2 the Airy function F should be real, we obtain: for real and distinct roots of Eq. (2.5) $f_\alpha^*(z_\alpha^*) = \overline{f_\alpha(z_\alpha)}$, and for complex-conjugate roots $f_\alpha^*(z_\alpha^*) = \overline{f_{3-\alpha}(z_{3-\alpha})}$.

Substituting (2.7) into (2.3) we obtain a representation for the components of the stress tensor

$$\sigma_{11} = -2\,\mathrm{Re}\sum_{\alpha=1}^{2}\mu_\alpha^2\left\{\Phi'_\alpha(z_\alpha)+\frac{1}{2}\eta\kappa_\alpha[p-Q_\alpha]\right\} \tag{2.8}$$

$$\sigma_{22} = 2\,\mathrm{Re}\sum_{\alpha=1}^{2}\left\{\Phi'_\alpha(z_\alpha)-\frac{1}{2}\eta\kappa_\alpha[p+Q_\alpha]\right\} \tag{2.9}$$

$$\sigma_{12} = 2\,\mathrm{Im}\sum_{\alpha=1}^{2}\mu_\alpha\left\{\Phi'_\alpha(z_\alpha)+\frac{1}{2}\eta\kappa_\alpha Q_\alpha\right\} \tag{2.10}$$

$$Q_\alpha = \frac{\partial}{\partial z_\alpha^*}\int_{z_{1\alpha}}^{z_\alpha} d\xi_\alpha p(\xi_\alpha, z_\alpha^*) \tag{2.11}$$

Here $\Phi_\alpha(z_\alpha) = f'_\alpha(z_\alpha)$ are analytic functions.

From (2.9) and (2.10), using the equalities

$$\kappa = \sum_{\alpha=1}^{2}\kappa_\alpha = \frac{b}{a_1} = \frac{(1-\nu)(\eta'/\eta+\nu')}{1-\nu'^2\Xi}, \qquad \sum_{\alpha=1}^{2}\kappa_\alpha\mu_\alpha^2 = 1 \tag{2.12}$$

we express the real and imaginary parts of the analytic functions Φ'_α in terms of the stress tensor and the pore pressure

$$2\,\mathrm{Re}\sum_{\alpha=1}^{2}\Phi'_\alpha(z_\alpha) = \sigma_{22}+\kappa\eta p+\eta\,\mathrm{Re}\sum_{\alpha=1}^{2}\kappa_\alpha Q_\alpha \tag{2.13}$$

$$2\,\mathrm{Im}\sum_{\alpha=1}^{2}\mu_\alpha\Phi'_\alpha(z_\alpha) = \sigma_{12}-\eta\,\mathrm{Im}\sum_{\alpha=1}^{2}\mu_\alpha\kappa_\alpha Q_\alpha \tag{2.14}$$

We also find a representation of the displacement field of a poroelastic medium in terms of the same analytic functions Φ'_α.

Substituting (2.8)–(2.10) into (2.1) and using the relations

$$\varepsilon_{22} = \frac{\partial}{\partial x_2}u_2, \qquad \frac{\partial}{\partial x_2} = i\mu_\alpha\left(\frac{\partial}{\partial z_\alpha}-\frac{\partial}{\partial \bar{z}_\alpha}\right) \tag{2.15}$$

we obtain the displacement field

$$u_2 = 2\,\mathrm{Im}\sum_{\alpha=1}^{2}q_\alpha\left\{\Phi_\alpha(z_\alpha)+\frac{1}{2}\eta\kappa_\alpha Q_\alpha\right\}, \qquad q_\alpha = (\mu_\alpha E')^{-1}[(1+\nu)\nu'\mu_\alpha^2+1-\nu'^2\Xi]$$

The partial derivative of the displacement field with respect to x_1

$$\frac{\partial}{\partial x_1}u_2 = 2\,\mathrm{Im}\sum_{\alpha=1}^{2}q_\alpha\left\{\Phi'_\alpha(z_\alpha)+\frac{1}{2}\eta\kappa_\alpha Q_\alpha\right\}, \qquad \frac{\partial}{\partial x_1} = \frac{\partial}{\partial z_\alpha}+\frac{\partial}{\partial z_\alpha^*}; \qquad \alpha = 1,2 \tag{2.16}$$

is needed below.

Specifying the load on the upper and lower edges of the crack, we obtain a Dirichlet problem in the exterior of a cut for two analytic functions $\Phi'_\alpha(z_\alpha)$ ($\alpha = 1, 2$) (2.14) and (2.15). Using the superposition principle we represent the stress and displacement fields in the form of sums of two fields: one of them corresponds to a continuous body under the action of loads applied with the body (σ_∞ is a uniform compressive stress and p_∞ is the unperturbed pressure of the permeating fluid), and the second to a body with a slit along the surfaces to which loads are applied. Here the boundary conditions at the crack edges have the form

$$\sigma_{22}^{\pm} = \sigma_\infty - p(x_1, t), \quad \sigma_{12}^{\pm} = 0, \quad x_2 = 0 \pm 0 \tag{2.17}$$

In order to solve the boundary-value problem (2.12), (2.13) it is also necessary to specify the values of the function $\mathrm{Im}\,Q_\alpha$ along the crack edges $x_2 = 0 \pm 0$.

It can be shown that at the crack edges Re $Q_\alpha = 0$ ($\alpha = 1, 2$). In particular, in the axisymmetric case this will be demonstrated below.

Consequently, the boundary-value problem (2.14), (2.15), (2.17) for the functions $\Phi'_\alpha(z_\alpha)$ ($\alpha = 1, 2$) can be written in the form

$$2\,\mathrm{Re}\left[\sum_{\alpha=1}^{2}\Phi'_\alpha(z_\alpha)\right]^{\pm} = -\Sigma^{\pm}(x_1,t),\quad 2\,\mathrm{Im}\left[\sum_{\alpha=1}^{2}\mu_\alpha\Phi'_\alpha(z_\alpha)\right]^{\pm} = \eta T^{\pm}(x_1,t),\quad |x_1|<l \tag{2.18}$$

$$\Sigma^{\pm}(x_1,t) = p(x_1,t)-\sigma_\infty - \eta\kappa(p(x_1,t)-p_\infty),\quad T^{\pm}(x_1,t) = -\mathrm{Im}\sum_{\alpha=1}^{2}\mu_\alpha\kappa_\alpha Q_\alpha^{\pm}$$

The solution of the Dirichlet boundary-value problem (2.18) for a slit can be obtained by standard methods [19] and has the form

$$\Phi'_\alpha(z) = -\frac{\mu_{3-\alpha}}{2\pi i(\mu_{3-\alpha}-\mu_\alpha)\sqrt{z^2-l^2}}\int_{-l}^{l}\frac{\sqrt{\zeta^2-l^2}\,\Sigma(\zeta,t)d\zeta}{\zeta - z} -$$

$$-\frac{\eta\kappa}{2\pi(\mu_{3-\alpha}-\mu_\alpha)}\int_{-l}^{l}\frac{T(\zeta,t)d\zeta}{\zeta-z}+\frac{\mu_{3-\alpha}}{(\mu_{3-\alpha}-\mu_\alpha)}\frac{C_0}{\sqrt{l^2-z^2}},\quad T=\frac{T^+-T^-}{2} \tag{2.19}$$

(C_0 is a constant).

Substituting the functions $\Phi'_\alpha(z_\alpha)$ ($\alpha = 1, 2$) into (2.16), using the Sokhotskii–Plemel formula and integrating over x_1, we obtain the crack opening in the transversely-isotropic poroelastic medium

$$w(x_1,t) = \frac{2[1-\nu'^2\Xi](\mu_1+\mu_2)}{\pi E'\mu_1\mu_2}\left\{\int_{x_1}^{l}\int_{0}^{\xi}\frac{\Sigma(\zeta,t)\xi d\zeta d\xi}{\sqrt{\xi^2-\zeta^2}\sqrt{\xi^2-x_1^2}}+\pi\eta\sum_{\alpha=1}^{2}\kappa_\alpha\int_{x_1}^{l}d\xi T_\alpha(\xi,t)\right\},$$

$$T_\alpha = \frac{T_\alpha^+ - T_\alpha^-}{2},\quad T_\alpha^{\pm} = \mathrm{Im}\,Q_\alpha^{\pm} \tag{2.20}$$

Here the Barenbatt fracture criterion [2] for a transversely-isotropic medium can be written in the form

$$\int_{0}^{l}\frac{\Sigma(\zeta,t)\zeta^n d\zeta}{\sqrt{l^2-\zeta^2}} = \frac{K_l}{\sqrt{2l}} \tag{2.21}$$

where K_l is the adhesion coefficient, and for a plane crack $n = 0$, while for an axisymmetric crack $n = 1$. Because the parameter η occurs in $\Sigma(\zeta, t)$ the resulting fracture criterion differs from the corresponding criterion for an elastic body.

In the limit as $\eta \to 0$ formula (2.20) gives the crack opening in the transversely-isotropic elastic body. If we also have $\mu_1 = \mu_2 = 1$, then (2.21) reduces to Sneddon's formula for an isotropic elastic body.

The second term in expression (2.20) gives the non-local contribution to the crack opening from the pressure distribution p of the fluid permeating the medium. The problem of calculating this term in the axisymmetric case will be considered in Section 6.

In such an approach one can thus integrate those equations of the coupled theory of transversely-isotropic poroelasticity which describe the deformation of the body. Here the problem of hydraulic fracture reduces to the pore pressure transfer equations (1.3) and (1.4) and a functional relating the crack opening (2.20) to the pore pressure. The fluid transfer equations, after transformation, can be reduced to a single diffusion-type equation for the pore pressure only with non-local sources associated with the change in the permeability of the medium when it is deformed.

3. THE AXISYMMETRIC PROBLEM

To describe the axisymmetric deformation of a porous medium saturated with fluid and a fluid filtering through it one uses the coupled consolidation equations (1.1) and (1.2) in a cylindrical system of coordinates ($i, j, k = r, \varphi, z$).

In the plane strained state an Airy function was introduced and the solution was then found using analytic function theory, but this approach is inapplicable to the three-dimensional axisymmetric problem. Hence for the three-dimensional poroelastic problem we proceed as follows: we write the equilibrium equations in displacements, and then solve them using the theory of generalized analytic functions.

The equilibrium equations in displacements have the form

$$\left[A_{44}\frac{\partial^2}{\partial z^2}+A_{11}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right]w_1+(A_{13}+A_{44})\frac{\partial^2}{\partial z\partial r}w_2=b_1\frac{\partial}{\partial r}p$$

$$(A_{13}+A_{44})\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial}{\partial z}w_1+\left[A_{33}\frac{\partial^2}{\partial z^2}+A_{44}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right]w_2=b_2\frac{\partial}{\partial r}p \tag{3.1}$$

$$b_1=\frac{2(1-\nu)}{E}\eta\left(A_{11}+A_{12}+A_{13}\Xi\frac{\eta'}{\eta}\right)\quad b_2=\frac{2(1-\nu)}{E'}\eta'\left(A_{33}+2A_{13}\frac{\eta}{\eta'\Xi}\right)$$

where w_1 is the radial displacement, and w_2 is the displacement along the direction of the axis of symmetry of the problem.

Unlike the equilibrium equations for an elastic medium, the equilibrium equations for a poroelastic body (3.1) contain derivatives of the pore pressure and so the method of solving these equations presented in [20] cannot be directly applied to this case.

We introduce the notation

$$D=A_{13}+A_{44},\quad A_j=A_{33}-A_{44}\mu_j^{-2},\quad B_j=A_{11}\mu_j^{-2}-A_{44};\quad j=1,2 \tag{3.2}$$

where μ_j are the roots of the characteristic equation

$$D^2\mu_j^{-2}=A_jB_j \tag{3.3}$$

Multiplying the first equation in (3.1) by B_1D and the second by A_1D, we transform them to the form

$$A_{11}B_1\frac{\partial}{\partial r}G_1+A_{44}D\frac{\partial}{\partial z}G_2=0,\quad A_{33}D\frac{\partial}{\partial z}G_1-A_1A_{44}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)G_2=C\frac{\partial}{\partial z}p \tag{3.4}$$

$$G_1=-\frac{Db_1}{A_{11}}p+D\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)w_1+A_1\frac{\partial}{\partial z}w_2,\quad G_2=B_1\frac{\partial}{\partial z}w_1-D\frac{\partial}{\partial r}w_2 \tag{3.5}$$

$$C=D(b_2A_1-b_1DA_{33}/A_{11})$$

Solving in turn first system (3.4), and then (3.5), we find the displacements w_1 and w_2 in the poroelastic medium. The choice of the coefficient in front of the pore pressure p in (3.5) enables one to eliminate $\partial p/\partial r$ from system (3.4), which is a necessary condition for applying generalized analytic function theory.

We represent Eqs (3.4) in the form

$$\frac{\partial}{\partial r}\left[\frac{\partial}{\partial z}(p_2U_2)\right]=-\mu_2^{-1}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(p_2V_2)\right]$$

$$\mu_2^{-1}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(p_2U_2)\right]=\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left[\frac{\partial}{\partial z}(p_2V_2)\right]+\eta_2\frac{\partial}{\partial z}p \tag{3.6}$$

$$p_2\frac{\partial}{\partial z}U_2=\frac{G_1}{A_{44}(\mu_2^{-2}-\mu_1^{-2})},\quad p_2\frac{\partial}{\partial z}V_2=\frac{DG_2}{A_{11}B_1\mu_2^{-2}(\mu_2^{-2}-\mu_1^{-2})}$$

$$\eta_2=CD=[\mu_2A_{11}A_1B_1A_{44}(\mu_2^{-2}-\mu_1^{-2})]^{-1}$$

We introduce complex variables, the variables conjugate to them, and derivatives with respect to those variables by the formulae

$$t_\alpha = \mu_\alpha z + ir, \quad t_\alpha^* = \mu_\alpha z - ir, \quad \alpha = 1,2 \tag{3.7}$$

$$2\frac{\partial}{\partial t_\alpha} = \mu_\alpha^{-1}\frac{\partial}{\partial z} - i\frac{\partial}{\partial r}, \quad 2\frac{\partial}{\partial t_\alpha^*} = \mu_\alpha^{-1}\frac{\partial}{\partial z} + i\frac{\partial}{\partial r}$$

We first multiply the first equation in (3.6) by i and add it to the second equation, and then multiply the first equation in (3.6) by $-i$ and also add it to the second, and then using (3.7) we obtain

$$2\frac{\partial}{\partial t_2^*}\psi_2 - \frac{\psi_2 - \psi_2^*}{t_2 - t_2^*} = \eta_2\frac{\partial}{\partial z}p, \quad 2\frac{\partial}{\partial t_2}\psi_2^* - \frac{\psi_2 - \psi_2^*}{t_2 - t_2^*} = \eta_2\frac{\partial}{\partial z}p \tag{3.8}$$

where $\psi_2 = (\partial/\partial z)(p_2\Lambda_2)$, $\psi_2^* = (\partial/\partial z)(p_2\Lambda_2^*)$ and $\Lambda_2 = U_2 + iV_2$, $\Lambda_2^* = U_2 - iV_2$.
Integrating (3.8) we find

$$\psi_2 = \chi_2 + \frac{1}{2}\eta_2\frac{\partial}{\partial z}\int_{t_{20}^*}^{t_2^*} p\,d\xi_2^*, \quad \psi_2^* = \chi_2^* + \frac{1}{2}\eta_2\frac{\partial}{\partial z}\int_{t_{20}}^{t_2} p\,d\xi_2 \tag{3.9}$$

Here χ_2 and χ_2^* are arbitrary generalized analytic functions (i.e. functions satisfying the equations $2\partial\chi_2/\partial t^*{}_2 - (\chi_2 - \chi_2^*)/(t_2 - t_2^*) = 0$ and $2\partial\chi_2^*/\partial t_2 - (\chi_2 - \chi_2^*)/(t_2 - t_2^*) = 0$ [20]).
Introducing in place of the generalized analytic functions χ_2 and χ_2^* the functions $\varphi_2 = \partial(p_2\chi_2)/\partial z$ and $\varphi_2^* = \partial(p_2\chi_2^*)/\partial z$ which are also generalized analytic functions, and integrating Eqs (3.9), we obtain

$$p_2(U_2 + iV_2) = p_2\varphi_2 + \frac{1}{2}\eta_2\int_{t_{20}^*}^{t_2^*} p\,d\xi_2^*, \quad p_2(U_2 - iV_2) = p_2^*\varphi_2 + \frac{1}{2}\eta_2\int_{t_{20}}^{t_2} p\,d\xi_2 \tag{3.10}$$

(the constant p_2 will be determined below).
Expressions (3.10) give the solution of the system of equations (3.4). Using these solutions we seek a solution of the linear system of equations (3.5) in the form of a superposition

$$\begin{Vmatrix} w_1 \\ w_2 \end{Vmatrix} = \begin{Vmatrix} w_{11} \\ w_{21} \end{Vmatrix} + \begin{Vmatrix} w_{12} \\ w_{22} \end{Vmatrix} \tag{3.11}$$

$$D\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)w_{11} + A_1\frac{\partial}{\partial z}w_{21} = G_1 + \alpha_1 p, \quad B_1\frac{\partial}{\partial z}w_{11} - D\frac{\partial}{\partial r}w_{21} = G_2 \tag{3.12}$$

$$D\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)w_{12} + A_1\frac{\partial}{\partial z}w_{22} = \alpha_2 p, \quad B_1\frac{\partial}{\partial z}w_{12} - D\frac{\partial}{\partial r}w_{22} = 0 \tag{3.13}$$

$$\alpha_1 + \alpha_2 = \frac{Db_1}{A_{11}}, \quad \alpha_1 = \frac{A_2 C}{A_{33}A_{44}D(\mu_2^{-2} - \mu_1^{-2})} \tag{3.14}$$

The solution of system (3.12) is given by the expressions

$$w_{11} = -p_2\omega_2 V_2, \quad w_{21} = p_2 U_2, \quad \omega_2 = D(\mu_2 B_2)^{-1} \tag{3.15}$$

The system of equations (3.13) is solved using expressions (3.7) and (3.14) in the same way as system (3.12)

$$w_{12} = -p_1\omega_1 V_1, \quad w_{22} = p_1 U_1 \tag{3.16}$$

$$p_1(U_1 + iV_1) = p_1\varphi_1 + \frac{1}{2}\eta_1\int_{t_{10}^*}^{t_1^*} p\,d\xi_1^*, \quad p_1(U_1 - iV_1) = p_1\varphi_1^* + \frac{1}{2}\eta_1\int_{t_{10}}^{t_1} p\,d\xi_1$$

$$\eta_1 = \frac{B_1\alpha_2\mu_1}{D^2}, \quad \omega_1 = D(\mu_1 B_1)^{-1}, \quad \alpha_2 = \frac{Db_1}{A_{11}} - \alpha_1$$

(φ_1 and φ_1^* are generalized analytic functions and the constant p_1 is determined below).
Because it follows from (3.11), (3.15) and (3.16) that

$$w_1 = -p_1\omega_1 V_1 - p_2\omega_2 V_2, \quad w_2 = p_1 U_1 + p_2 U_2 \tag{3.17}$$

we impose evenness conditions on the functions U_j, V_j ($j = 1, 2$), which follow from the symmetry of the problem

$$U_j(z,r) = U_j(z,-r), \quad V_j(z,r) = -V_j(z,-r)$$

Hence it follows, using (3.10), that $\varphi_j(t_j) = \varphi_j^*(t_j^*)$,
We consider the two possible cases for the roots of Eq. (3.3)
1. $\operatorname{Im}\mu_j = 0$, so that $U_j = \overline{U}_j$, $V_j = \overline{V}_j$, i.e. $\varphi^*_j(\underline{t}_j) = \overline{\varphi_j(t_j)}$;
2. $\operatorname{Im}\mu_j \neq 0$, $\mu_1 = \bar{\mu}_2$, so that $U_j = \overline{U}_j$, $V_{3-j} = \overline{V}_{3-j}$, i.e. $\varphi^*_{3-j}(t_{3-j}) = \overline{\varphi_{3-j}(t_{3-j})}$.
Thus using the evenness conditions, from (3.10), (3.15) and (3.16) we obtain

$$w_1 = \operatorname{Re}\sum_{\alpha=1}^{2}\{i\omega_\alpha p_\alpha \varphi_\alpha - I_\alpha^-\}, \quad w_2 = \operatorname{Re}\sum_{\alpha=1}^{2}\{p_\alpha\varphi_\alpha + I_\alpha^+\} \tag{3.18}$$

$$I_\alpha^\pm = \frac{i}{4}\eta_\alpha\left[\int_{t_{\alpha 0}}^{t_\alpha} p\,d\xi_\alpha \pm \int_{t_{\alpha 0}^*}^{t_\alpha^*} p\,d\xi_\alpha^*\right]$$

We put $p_i = q_i$. (Here and below we use the notation introduced in Section 3.) We express the stresses in terms of the deformations from (1.2), and then using (3.18) and the fact that the roots of the characteristic polynomial are either real or complex-conjugate, we obtain a representation for the components of the stress tensor in terms of two generalized analytic functions

$$\operatorname{Re}\sum_{\alpha=1}^{2}\varphi'_\alpha(z_\alpha) = \sigma_{zz} + \kappa\eta p + \eta\operatorname{Re}\sum_{\alpha=1}^{2}\kappa_\alpha Q_\alpha \tag{3.19}$$

$$\operatorname{Im}\sum_{\alpha=1}^{2}\mu_\alpha\varphi'_\alpha(z_\alpha) = \sigma_{rz} - \eta\operatorname{Im}\sum_{\alpha=1}^{2}\mu_\alpha\kappa_\alpha Q_\alpha \tag{3.20}$$

$$Q_\alpha = \frac{\partial}{\partial t_\alpha^*}\int_{t_{0\alpha}}^{t_\alpha} d\xi_\alpha p(\xi_\alpha, t_\alpha^*)$$

To change from the generalized analytic functions φ_α ($\alpha = 1, 2$) to ordinary analytic functions we use the integral operators S_i^{-1} and S_i ($i = 1, 2$) [20].
Thus, if φ is a generalized analytic function, then $\Phi = S^{-1}\varphi$ is an ordinary analytic function.
The contour of integration for the operators S_i^{-1} and S_i is chosen to be the straight line perpendicular to the x axis and passing through the point $z = z_0$. In this case the operators S_i^{-1} and S_i do not depend on the index i, i.e. $S_i^{-1} = S^{-1}$ and $S_i = S$, and when φ_i and Φ_i satisfy the evenness conditions, they can be written in the form

$$\Phi_i(t_i) = S^{-1}\varphi_i = \operatorname{sign}(y)\frac{d}{dy}\int_0^y [r\operatorname{Re}\varphi_i(\tau_i) + iy\operatorname{Im}\varphi_i(\tau_i)]\frac{dr}{\sqrt{y^2 - r^2}} = s_0^{-1}\operatorname{Re}\varphi_i + is_1^{-1}\operatorname{Im}\varphi_i$$

$$\varphi_i(t_i) = S\Phi_i = \frac{2}{\pi r}\int_0^r [r\operatorname{Re}\Phi_i(\sigma_i) + iy\operatorname{Im}\Phi_i(\sigma_i)]\frac{dy}{\sqrt{r^2 - y^2}} = s_0\operatorname{Re}\Phi_i + is_1\operatorname{Im}\Phi_i \tag{3.21}$$

$$(\tau_i = t_i = \mu_i z + ir, \quad x = z = z_0, \quad \zeta_i = \sigma_i = \mu_i x + iy).$$

Using the operators S_k^{-1} ($k = 0, 1$) we change in representations (3.19) and (3.20) from generalized analytic functions to ordinary analytic functions

$$\operatorname{Re}\sum_{\alpha=1}^{2}\Phi'_\alpha(z_\alpha) = s_0^{-1}\left[\sigma_{zz} + \kappa\eta p + \eta\operatorname{Re}\sum_{\alpha=1}^{2}\kappa_\alpha Q_\alpha\right] \tag{3.22}$$

$$\operatorname{Im}\sum_{\alpha=1}^{2}\mu_\alpha\Phi'_\alpha(z_\alpha) = s_1^{-1}\left[\sigma_{rz} - \eta\operatorname{Im}\sum_{\alpha=1}^{2}\mu_\alpha\kappa_\alpha Q_\alpha\right]$$

We will formulate a mixed boundary-value problem for the analytic functions $\Phi'_\alpha(z_\alpha)$ ($\alpha = 1, 2$), using the super-position principle, i.e. representing the stress and displacement fields in the form of the sum of two fields, one of which corresponds to a continuous body with loads applied inside the body (σ_∞ being a homogeneous compressive stress and p_∞ being the unperturbed pressure of the permeating fluid), and the second to a body with a cut with applied surface loads

$$\sigma_{zz} = p - \sigma_\infty, \quad \sigma_{rz} = 0 \tag{3.23}$$

The operators s_k^{-1} ($k = 0, 1$) relate the three-dimensional axisymmetric strained state (z, r) of the plane state symmetric abut the axis $x = 0$ ($y = z$, $r \to x$). Bearing in mind that the value of the radicals in the operators s_k^{-1} have opposite signs at the two edges of the cut, and setting the constants $\tau_{\alpha 0}$, $t^*_{\alpha 0}$ ($\alpha = 1, 2$) to zero, to satisfy the symmetry conditions (about the $x = 0$ axis) we obtain from (3.22) that

$$\mathrm{Re}\left[\sum_{\alpha=1}^{2} \Phi'_\alpha(z_\alpha)\right]^{\pm} = [s_0^{-1}\Sigma(r,t)]^{\pm}, \quad \mathrm{Im}\left[\sum_{\alpha=1}^{2} \mu_\alpha \Phi'_\alpha(z_\alpha)\right]^{\pm} = \eta[s_1^{-1}T(r,t)]^{\pm} \tag{3.24}$$

$$|x_1| < l, \quad \Sigma^{\pm}(r,t) = p(r,t) - \sigma_\infty - \eta\kappa(p(r,t) - p_\infty), \quad T^{\pm}(r,t) = -\mathrm{Im}\sum_{\alpha=1}^{2} \mu_\alpha \kappa_\alpha Q_\alpha^{\pm}$$

Here we have used $\eta\ \mathrm{Re}\ \Sigma_{\alpha=1}^2\ \kappa_\alpha Q_\alpha = 0$, which we will justify later.

The solution of the Dirichlet boundary-value problem for the cut (3.24) is similar to (2.19).

Differentiating expression (3.18) with respect to and changing from the generalized analytic functions $\varphi'_\alpha(z_\alpha)$ to analytic functions $\Phi'_\alpha(z_\alpha)$ using the operators s_k and s_k^{-1} ($k = 1, 2$), we substitute into this expression the functions $\varphi'_\alpha(z_\alpha)$ found from the solution of boundary-value problem (3.24). Then, using the Sokhotskii–Plemel formula and integrating over r, we find the crack opening

$$w(r,t) = \frac{2(1-\nu'^2\Xi)(\mu_1+\mu_2)}{\pi E'\mu_1\mu_2}\left\{\int_r^l \int_0^\xi \frac{\Sigma(\zeta,t)\zeta d\zeta d\xi}{\sqrt{\xi^2-\zeta^2}\sqrt{\xi^2-r^2}} + \pi\eta\sum_{\alpha=1}^{2}\kappa_\alpha W_\alpha\right\} \tag{3.25}$$

$$W_\alpha = \int_r^l T_\alpha(\xi,t)d\xi, \quad T_\alpha = (T_\alpha^+ - T_\alpha^-)/2, \quad T_\alpha^{\pm} = \mathrm{Im}\,Q_\alpha^{\pm}$$

Here we have used the Barenblatt fracture criterion (2.21) for a transversely-isotropic medium.

All the conclusions in Section 2 concerning fracture criteria for various limiting cases and the transfer equations for the pore pressure in the plane strained state remain true for the axisymmetric problem.

In the following sections we obtain the solution of the steady axisymmetric problem and we are shown how a real variable can be obtained in the second term of formula (3.25).

4. THE STEADY SOLUTION

We consider the steady hydrofracture problem in a transversely-isotropic porous fluid-saturated medium with an axisymmetric non-moving crack of radius l = const. Here it is assumed that the pore pressure of the fluid depends only on coordinates r and z ($p(r, \varphi, z, t) = p(r, z)$). Here the fluid transfer equation in a transversely-isotropic porous medium (1.3), (1.4) in cylindrical coordinates reduces to a Laplace equation for the pore pressure

$$\frac{k_1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}p\right) + k_2\left(\frac{\partial^2}{\partial z^2}p\right) = 0 \tag{4.1}$$

Because of the symmetry of the problem about the $z = 0$ plane we will formulate the boundary-value problem for the Laplace equation (4.1) in the upper half-plane $z > 0$

$$p(r, z=0) - p_\infty = p_0(r) - p_\infty, \quad 0 \le r < l; \quad \frac{\partial}{\partial z}p(r, z=0) = 0, \quad r > l \tag{4.2}$$

$$p(r, z \to \infty) - p_\infty = 0$$

Boundary-value problem (4.1), (4.2) in cylindrical coordinates can be solved by the method of dual integral equations [21].

We will consider the special case of an ideal crack, i.e. a crack with a high hydraulic conductivity. The pressure distribution for fluid in such a crack can be taken to be approximately constant along its edges

$$p_0(r) = p_0 = \text{const} \tag{4.3}$$

The solution of boundary-value problem (4.1), (4.2) with (4.3) has the form

$$p(r,z) = \frac{2}{\pi}(p_0 - p_\infty)\arcsin\left(\frac{2l}{\rho^+ + \rho^-}\right), \quad \rho^\pm = \sqrt{(l \pm r)^2 + \chi^2 z^2}, \quad \chi = \sqrt{\frac{k_1}{k_2}} \tag{4.4}$$

Changing to complex variables t_j, t_j^* (3.7) in expression (4.4), we obtain

$$p(r,z) = p\left(\frac{t_j - t_j^*}{2i}, \frac{t_j + t_j^*}{2\mu_j}\right) = \tilde{p}(t_j, t_j^*) = \frac{2}{\pi}(p_0 - p_\infty)\arcsin Z \tag{4.5}$$

$$Z = 2l[f(t_j, t_j^*) + f(t_j^*, t_j)]^{-1}$$

$$f(u, \upsilon) = ([l - i(1-\alpha_j)u + i\alpha_j\upsilon][l + i(1-\alpha_j)\upsilon - i\alpha_j u])^{1/2}, \quad \alpha_j = (\mu_j^2 + \chi^2)/(2\mu_j^2)$$

The tilde over the pressure $\tilde{p}(t_j, t_j^*)$ is henceforth omitted.

The hydraulic fracture opening in a transversely-isotropic poroelastic medium can be obtained by substituting (4.5) into (3.25)

$$w(r) = \frac{2[1 - \nu'^2\Xi](\mu_1 + \mu_2)}{\pi E'\mu_1\mu_2}\left\{l[p_0 - \sigma_\infty + \eta\kappa(p_0 - p_\infty)]\sqrt{1 - \left(\frac{r}{l}\right)^2} + \pi\eta\sum_{\alpha=1}^{2}\kappa_\alpha W_\alpha\right\} \tag{4.6}$$

The functions W_α are determined in terms of elementary functions, but in the general case they are rather complicated and their form depends on the parameters a_j, i.e. on the roots of the characteristic equation (3.3) μ_j $(j = 1, 2)$ and the ratio $\chi = \sqrt{(k_1/k_2)}$. Hence we give the function W_α only for real roots μ_j of the characteristic equation, and consequently for real α_j.

If $\sqrt{1/2} \leqslant \alpha_j, \leqslant 1$ we have

$$\begin{aligned} W_\alpha = &-\frac{2\alpha_j - 1}{\sqrt{\beta_j}}\left\{\frac{\pi}{2} - \frac{r}{l}\operatorname{arctg}\left(\frac{2\sqrt{\beta_j}}{2\alpha_j - 1}\frac{r}{\sqrt{l^2 - r^2}}\right) - \right. \\ &- \frac{2\alpha_j - 1}{\sqrt{1 - 8\beta_j}}\operatorname{arcctg}\left(\frac{2\sqrt{\beta_j}}{\sqrt{1 - 8\beta_j}}\frac{l}{\sqrt{l^2 - r^2}}\right) - \\ &- \sum_{k=1}^{2}\left[\operatorname{arctg}\sqrt{\frac{1 + k - \alpha_j}{1 - k + \alpha_j}} - \frac{r}{l}\operatorname{arctg}\sqrt{\frac{(1-\alpha_j)\alpha_{kj}^-}{\alpha_j[l + (-1)^k\alpha_j r]}} - \right. \\ &- \frac{1}{2(2\alpha_j - 1)}\left(\arcsin\left([4(k - \alpha_j)^2 - 1]\sqrt{1 - (k - \alpha_j)^2}\right) - \right. \\ &\left.\left.\left. - \arcsin\left(\frac{(-1)^k(2\alpha_j - 1)l + 2\beta_j r}{l}\right) - \arcsin\left(\frac{b_{kj}}{a_{kj}^+}\right)\right)\right]\right\} \end{aligned} \tag{4.7}$$

$$\beta_j = \alpha_j(1 - \alpha_j), \quad a_{kj}^\pm = l \pm (-1)^k(1 - \alpha_j)r, \quad b_{kj} = (-1)^k(2\alpha_j - 1)l - (1 - 2\beta_j)r$$

In the limit as $\alpha_j \to 1$ $(\mu_j \to \chi)$ we obtain the crack opening in an isotropic poroelastic medium

$$w(r)=\frac{2(1-\nu^2)}{\pi G}l\left\{[p_0-\sigma_\infty+\eta(p_0-p_\infty)]\sqrt{1-\left(\frac{r}{l}\right)^2}+\right.$$

$$\left.+\eta(p_0-p_\infty)\left[\sqrt{2}-\sqrt{1+\frac{r}{l}}-\sqrt{1-\frac{r}{l}}+\sqrt{1-\left(\frac{r}{l}\right)^2}\right]\right\} \tag{4.8}$$

If $\alpha_j < \sqrt{1/2}$ the expression $8\alpha_j^2 - 8\alpha_j + 1$ in (4.7) becomes negative, and in this case, considering $\operatorname{arcctg} z$ and $\sqrt{z}$ as functions of complex variables and changing from $\operatorname{arcctg} z$ to a logarithmic function, formula (4.7) can be transformed to a real expression. The range of variation of a_j ($j = 1, 2$) that we are considering corresponds to roots of the characteristic equation which satisfy the inequality $\mu_j > \chi$ ($j = 1, 2$).

One can proceed similarly when $\alpha_j > 1$, i.e. $\mu_j < \chi$. Here it is also necessary to consider the function in formula (4.7) as a function of complex variables, and after appropriate transformations it can also be reduced to a function of a real variable.

5. CALCULATION OF THE FUNCTION T_α

In the general case when solving a non-stationary hydrofracture problem for a transversely-isotropic poroelastic medium it becomes necessary to construct the functions $T_\alpha = \operatorname{Im}(Q_\alpha^+ - Q_\alpha^-)/2$ ($\alpha = 1, 2$) in terms of the pore pressure $p(r, z, t)$, which is a real function of the real variables r and z. To this end we change from the variables r and z to complex variables t_j and t_j^* (3.7). Then

$$Q_\alpha=\frac{\partial}{\partial t_\alpha}\int_0^{t_\alpha^*}p\left(\frac{t_\alpha-\tau}{2i},\ \frac{t_\alpha+\tau}{2\mu_\alpha}\right)d\tau=\frac{1}{2i}\int_0^{t_\alpha^*}U\left(\frac{t_\alpha-\tau}{2i},\ \frac{t_\alpha+\tau}{2\mu_\alpha}\right)d\tau+$$

$$+\frac{1}{2\mu_\alpha}\int_0^{t_\alpha^*}V\left(\frac{t_\alpha-\tau}{2i},\ \frac{t_\alpha+\tau}{2\mu_\alpha}\right)d\tau \tag{5.1}$$

$$(U=\partial p/\partial r,\ \ V=\partial p/\partial z)$$

The boundary values of function (5.1) at $z = 0 \pm 0$, using the properties of Cauchy integrals, are taken along the infinite line $z = 0$ [19], and also the condition $U = V = 0$ as $r^2 + z^2 \to \infty$, can be represented in the form

$$Q_\alpha^\pm=-\frac{1}{2\pi^2}\int_0^r d\tau\int_{-\infty}^{\infty}d\xi\int_{\infty}^{\infty}d\zeta\frac{\mu_\alpha U(\xi,\zeta)\pm iV(\xi,\zeta)}{[2\xi\pm(r-\tau)][2\mu_\alpha\zeta-i(r+\tau)]} \tag{5.2}$$

In obtaining formula (5.2) we used symmetry properties of the functions U and V, associated with the symmetry conditions of the problem: $U(-\xi, \zeta) = -U(\xi, \zeta)$, $U(\xi, -\zeta) = U(\xi, \zeta)$, $V(-\xi, \zeta) = V(\xi, \zeta)$, $V(\xi, -\zeta) = -V(\xi, \zeta)$.

From (5.2), the symmetry properties of U and V, and the fact that the roots of the characteristic equation are real or complex-conjugate, it follows that

$$\operatorname{Re}\sum_{\alpha=1}^{2}\kappa_\alpha(Q_\alpha^+ + Q_\alpha^-)=0,\quad T_\alpha=\operatorname{Im}(Q_\alpha^+ - Q_\alpha^-)=-\frac{1}{4\pi^2}\int_0^\infty d\xi\int_0^\infty d\zeta\zeta V(\xi,\zeta)L_\alpha(r,\xi,\zeta)\} \tag{5.3}$$

$$L_\alpha(r,\xi,\zeta)=16\int_0^\infty d\tau\frac{r-\tau}{(2\xi)^2-(r+\tau)^2}\operatorname{Im}\left\{\frac{i\mu_\alpha}{(2\mu_\alpha\zeta)^2+(r+\tau)^2}\right\}$$

Thus, by obtaining for a given time t the velocity field $V = \partial p/\partial z$ in the domain $r \in [0, \infty)$, $z \in [0, \infty)$ from the transfer equation for a permeating fluid (1.3) and then computing the functions T_α from formulae (5.3), one can construct the crack opening from (3.25).

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